MATH GU 4156 **Advanced Probability Theory** Columbia University (Fall 2023)

Lecture 4: Filtrations and Stopping Times

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4.1 Filtrations

Consider a measurable space $(\Omega, \mathcal{F})$, where Ω is a non-empty "sample space" and $\mathcal{F}$ is a σ-algebra of subsets of Ω ("events"), as well as an increasing sequence

$$\{\emptyset, \Omega\} =: \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq$$

of σ-algebras of $\mathcal{F}$.

Typical situation: sequence $X_1, X_2, ...$ of (real-valued) $\mathcal{F}$-measurable functions on Ω, a.k.a. "random variables", successive outcomes of an experiment that we monitor on a day-to-day basis. Then

$$\mathcal{F}_n = \sigma(X_1, ..., X_n),$$

the smallest σ-algebra w.r.t. which the vector $(X_1, ..., X_n)$ is measurable, has the significance of information accumulated by day t=n.

We call **filtration** the collection

$$\mathbb{F} = \{F_n\}_{n \in \mathbb{N}_0}$$

and introduce also the "ultimate" σ-algebra

$$\mathcal{F}_\infty := \sigma\left(\cup_{n \in \mathbb{N}_0} \mathcal{F}_n, \right)$$

the information that "accrues all the way to the end of time".

Given a filtrated measurable space $(\Omega, \mathcal{F}), \mathbb{F}$ as just described, we say that a sequence of measurable functions $Y_1, Y_2...$ on $(\Omega, \mathcal{F})$ is

- **adapted to** $\mathbb{F}$, if Y_n is $\mathcal{F}_n$-measurable, $\forall n \in \mathbb{N}$.

In the "canonical" case, where $\mathbb{F}$ is generated by $\{X_n\}_{n\in\mathbb{N}}$, this means

$$Y_n = f_n(X_1, ..., X_n),\ \forall n \in \mathbb{N}$$

for some measurable $f_n : \mathbb{R}^n \to \mathbb{R}$.

- **predictable w.r.t.** $\mathbb{F}$if Y_n is $\mathcal{F}_{n-1}$-measurable, $\forall n \in \mathbb{N}$.

In the "canonical" case,

$$Y_n = g_n(X_1, ..., X_{n-1}),\ \forall n \in \mathbb{N}$$

for some measurable $g_n : \mathbb{R}^{n-1} \to \mathbb{R}$.

4.2 Stopping Time of $\mathbb{F}$

A measurable mapping $T : \Omega \to N_0 \cup +\infty$ with the property

$$\{T \leq n\} \in \mathcal{F}_n,\ \ \forall n \in \mathbb{N} \tag{4.1}$$

is called a **stopping time** of $\mathbb{F}$.

Exercise: (4.1) is equivalent to

$$\{T = n\} \in \mathcal{F}_n,\ \ \forall n \in \mathbb{N}.$$

Prime Example: First Hitting Time of $A \in \mathbb{B}(\mathbb{R})$

$$H_A := \begin{cases} \min\{n \in \mathbb{N} : Y_n \in A\} \\ \infty,\ \text{if } \{n \in \mathbb{N} : Y_n \in A\} = \emptyset \end{cases}$$

for some $Y_1, Y_2 ...$ adapted to $\mathbb{F}$. This is because

$$\{H_A = n\} = \{Y_1 \notin A, Y_2 \notin A, ..., Y_{n-1} \notin A, Y_n \in A\} \in \mathcal{F}_n,\ \ \forall n \in \mathbb{N}.$$

4.2.1 Algebra of Stopping Times

(i) if $S, T, \{T_n\}_{n\in\mathbb{N}}$ are stopping times, then so are

$$S+T,\ S\wedge T,\ S\vee T,\ \sup_n T_n,\ \inf_n T_n,\ \limsup_{n\to\infty} T_n,\ \liminf_{n\to\infty} T_n,$$

and of course $\lim_{n\to\infty} T_n$ whenever it exists.

(ii) The fixed time $T = m$ (some $m \in \mathbb{N}$) is a stopping time.

(iii) The difference $T - S$ is in general NOT a stopping time.

For instance, $H_A - 1$ is NOT a stopping time:

$$\{H_A - 1 = m\} = \{H_A = m+1\} = \{Y_1 \notin A, Y_2 \notin A, ..., Y_m \notin A, Y_{m+1} \in A\} \notin \mathcal{F}_n,$$

in general.

Likewise, the time

$$D_A := \begin{cases} \max\{n \in \mathbb{N}_0 : Y_n \in A\} \\ 0,\ \text{if } \{n \in \mathbb{N} : Y_n \in A\} = \emptyset \end{cases}$$

of last visit in a given set $A \in \mathbb{B}(\mathbb{R})$ by some $\mathbb{F}$-adapted sequence $Y_0, Y_1, Y_2...$ is in general NOT an $\mathbb{F}$-stopping time:

$$\{D_A = m\} = \{Y_m \in A, Y_{m+1} \notin A, Y_{m+2} \notin A, ...\} \notin \mathcal{F}_m,$$

in general.

4.2.2 σ-algebra of Events Revealed up until a Stopping Time T

For a given "date" (trivial stopping time) $T = m$, we have $\mathcal{F}_t = \mathcal{F}_m$ as the information accumulated up until T. Can we generalize this notion to arbitrary stopping times?

Yes, as follows:

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \le n\} \in \mathcal{F}_n, \forall n \in \mathbb{N}_0\} = \underline{\{A \in \mathcal{F} : A \cap \{T = n\} \in \mathcal{F}_n, \forall n \in \mathbb{N}_0\}\ \ \text{(Exercise)}}$$

Here are a few consequences of this definition, that should be checked very thoroughly.

1. The thus defined collection of events is a σ-algebra, with respect to which T is measurable.

2. $\mathcal{F}_T = \mathcal{F}_m$, if $T \equiv m$.

3. If $S \leq T$ are $\mathbb{F}$-stopping times, then

$$\mathcal{F}_S \subseteq \mathcal{F}_T.$$

4. For arbitrary stopping time S, T we have

$$F_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T.$$

5. The events $\{S < T\}, \{S > T\}, \{S = T\}$ belong to $F_{S \wedge T}$.

For any $\mathbb{F}$-adapted sequence $(Y_n)_{n \in \mathbb{N}_0} = \mathcal{Y}$ and ANY random variable $T : \Omega \to N_0 \cup \{+\infty\}$, we define

$$Y_T := \sum_{k \in \mathbb{N}_0} Y_k \mathbb{I}_{\{T=k\}} + (\limsup_{n \to \infty} Y_n) \mathbb{I}_{\{T=\infty\}}$$

"the value of the sequence at the random time T", as well as the new random sequence

$$Y_{T \wedge n}, n \in N_0$$

"the random sequence $\mathcal{Y}$, stopped at time T."

<u>Exercise:</u>

Check that, if T is a stopping time, then

(i) the random variable Y_T is $\mathcal{F}_T$-measurable,

(ii) the random sequence $(Y_{T \wedge n})_{n \ in \mathbb{N}_0}$ is adapted to $\mathbb{F}$.